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Dynamical aspects of the fuzzy CP^2 in the large N reduced model with a cubic term

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Abstract: "Fuzzy CP^2", which is a four-dimensional fuzzy manifold analogous to the fuzzy 2-sphere (S^2), appears as a classical solution in the dimensionally reduced 8d Yang-Mills model with a cubic term involving the structure constant of the SU(3) Lie algebra. Although the fuzzy S^2, which is also a classical solution of the same model, has actually smaller free energy than the fuzzy CP^2, Monte Carlo simulation shows that the fuzzy CP^2 is stable even nonperturbatively due to the suppression of tunneling effects at large N as far as the coefficient of the cubic term (α) is sufficiently large. As α is decreased, both the fuzzy CP^2 and the fuzzy S^2 collapse to a solid ball and the system is essentially described by the pure Yang-Mills model ($\alpha = 0$). The corresponding transitions are of first order. The gauge group generated dynamically above the critical point turns out to be of rank one for both CP^2 and S^2 cases. Above the critical point, we also perform perturbative calculations for various quantities *to all orders*, taking advantage of the one-loop saturation of the effective action in the large-N limit. By extrapolating our Monte Carlo results to $N = \infty$, we find excellent agreement with the all order results.

Keywords: Matrix Models, Non-Commutative Geometry, Nonperturbative Effects.

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1. Introduction

Fuzzy spheres [1], which are simple compact noncommutative manifolds, have been recently discussed extensively in the literature. One of the motivations comes from the general expectation that noncommutative geometry provides a crucial link to string theory and quantum gravity. Indeed Yang-Mills theories on noncommutative geometry appear in a certain low energy limit of string theory [2]. There is also an independent observation that the space-time uncertainty relation, which is naturally realized by noncommutative geometry, can be derived from some general assumptions on the underlying theory of quantum gravity [3]. Another motivation is to use fuzzy spheres as a regularization scheme alternative to the lattice regularization [4]. Unlike the lattice, fuzzy spheres preserve the continuous symmetries of the space-time considered, and hence it is expected that the situation concerning chiral symmetry [5 – 23] and supersymmetry might be improved.

As expected from the connection to string theory [24], fuzzy spheres appear as classical solutions in matrix models with a Chern-Simons-like term [25 – 29] and their dynamical properties have been studied in refs. [30 – 39]. One can actually use matrix models to define a regularized field theory on the fuzzy spheres as well as on a noncommutative torus [40], which enables nonperturbative studies of such theories from first principles [41]. These matrix models belong to the class of the so-called dimensionally reduced models (or large-N reduced models), which is widely believed to provide a constructive definition of superstring and M theories [42 – 44]. The space-time is represented by the eigenvalues of the bosonic matrices, and in the IIB matrix model [43], in particular, the dynamical generation of *four*-dimensional space-time (in *ten*-dimensional type IIB superstring theory) has been discussed by many authors [45 – 58].

In ref. [59] we have studied the dimensionally reduced 3d Yang-Mills-Chern-Simons (YMCS) model, which has the fuzzy 2-sphere (S^2) as a classical solution [26]. Unlike previous works we have performed nonperturbative first-principle studies by Monte Carlo simulation. We observed a first-order phase transition as we vary the coefficient (α) of the Chern-Simons term. For small α the large-N behavior of the model is the same as in the pure Yang-Mills model, whereas for large α a single fuzzy S^2 appears dynamically.

For obvious reasons it is interesting to extend this work to a matrix model which accommodates a *four*-dimensional fuzzy manifold. In ref. [60] we have studied the dimensionally reduced 5d Yang-Mills model with the quintic Chern-Simons term [27, 29], which is known to have the fuzzy 4-sphere (S^4) as a classical solution [61]. Unlike the fuzzy S^2 case, however, the fuzzy S^4 is unstable at the classical level, and Monte Carlo simulation confirmed that it does not stabilize even at the quantum level. The negative result is essentially due to the fact that the *quintic* Chern-Simons term has higher powers in A_μ than the Yang-Mills term.

This motivates us to return to the class of models with a *cubic* term. As a candidate of a four-dimensional fuzzy manifold, we study the fuzzy CP^2 [28, 34, 62 – 65], which appears as a classical solution in the dimensionally reduced 8d Yang-Mills model with a cubic term involving the structure constant of the SU(3) Lie algebra. In fact the fuzzy S^2, which is also a classical solution of this model, has smaller free energy than the fuzzy CP^2. Monte Carlo simulation shows, however, that the fuzzy CP^2 is stable even nonperturbatively due to the suppression of tunneling effects at large N as far as the coefficient (α) of the cubic term is sufficiently large. As we decrease α, both the fuzzy CP^2 and the fuzzy S^2 collapse to a solid ball, and the system is essentially described by the pure Yang-Mills model ($\alpha = 0$). The corresponding phase transitions are of first order, and the lower critical point agrees with the analytical result obtained from the one-loop effective action. Since the one-loop effective action is saturated at one loop in the large-N limit, the analytical result for the critical point is expected to be free from higher loop corrections.

Above the critical point, we perform perturbation calculations for various quantities *to all orders*, taking advantage of the one-loop saturation of the effective action in the large-N limit. This technique was originally proposed for a supersymmetric model [37], where the effective action is saturated only at two loop. In the bosonic case, one can obtain all order results by performing essentially the one-loop calculation [66]. By extrapolating our Monte Carlo results for various observables to $N=\infty$, we find excellent agreement with the all order results.

In the large-N reduced models, not only the space-time [45] but also the gauge group [67] is expected to appear dynamically. While there are certain evidences in the IIB matrix model that indeed *four-dimensional* space-time appears dynamically [53 − 55, 57, 68, 69], the issue of the gauge group is totally unclear. The models we are studying may be considered as a toy model in which one may obtain a definite answer to such a question, since the gauge group of rank k naturally appears if the true vacuum is given by k coincident fuzzy manifolds. The value of k should be determined dynamically, and the result may of course depend on the model one considers. In the present model we find the gauge group to be of rank one for both the fuzzy CP^2 and the fuzzy S^2. In arriving at this conclusion, the existence of the first-order phase transition plays a crucial role as in our previous work [59].

This paper is organized as follows. In section 2 we define the model and show that the fuzzy CP^2 and the fuzzy S^2 appear as classical solutions. In sections 3 and 4 we study the properties of the fuzzy CP^2 and the fuzzy S^2, respectively. In section 5 we compare the free energy for the fuzzy CP^2 and the fuzzy S^2 to discuss which is the true vacuum whenever they exist. In section 6 we study the properties of the k coincident fuzzy CP^2. In section 7 we determine the rank of the dynamical gauge group for the fuzzy CP^2 and the fuzzy S^2. Section 8 is devoted to a summary and discussions. In appendix A we present the explicit form of the fuzzy CP^2 configuration. In appendix B we formulate the perturbation theory around fuzzy manifolds and obtain an expression for the one-loop free energy. In appendices C and D we show the perturbative calculations for the fuzzy CP^2 and the fuzzy S^2, respectively.

2. The model and its classical solutions

The model we study is defined by the action

$$S = N\,\mathrm{tr}\left(-\frac{1}{4}\,[A_\mu, A_\nu]^2 + \frac{2}{3}\,i\,\alpha\, f_{\mu\nu\rho}\, A_\mu\, A_\nu\, A_\rho\right)\,, \tag{2.1}$$

where A_μ ($\mu = 1, \cdots, 8$) are $N\times N$ traceless hermitian matrices. Here and henceforth we sum over repeated indices. The coefficient $f_{\mu\nu\rho}$ is the structure constant of the SU(3) Lie algebra, whose nonzero components are given explicitly by

$$f_{123} = 1\,,\quad f_{458} = f_{678} = \frac{\sqrt{3}}{2}\,,\quad f_{147} = f_{246} = f_{257} = f_{345} = f_{516} = f_{637} = \frac{1}{2}\,. \tag{2.2}$$

The pure Yang-Mills model ($\alpha = 0$) and its obvious generalization to D dimensions with D matrices A_μ ($\mu = 1, \ldots, D$) have been studied by many authors. In particular, the large-N dynamics of the model have been studied by the $1/D$ expansion and Monte Carlo

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simulation [70]. The partition function was conjectured [71] and proved [72] to be finite for $N > D/(D-2)$. (See refs. [73 – 75] for the supersymmetric case.) The partition function in the presence of the Chern-Simons term has been studied analytically for $N = 2$ [76], and it turned out to be convergent in the supersymmetric case, but not in the bosonic case. It is also proved that adding a Myers term (the cubic term in the present case) does not affect the convergence as far as the original path integral converges absolutely [77], which means, in particular, that the partition function of our model is convergent for $N \geq 4$.

The classical equation of motion of the model (2.1) is given by

$$[A_\nu, [A_\mu, A_\nu]] - i\,\alpha\, f_{\mu\nu\rho}[A_\nu, A_\rho] = 0\,. \tag{2.3}$$

One can easily see that there exists a solution of the form

$$A_\mu = \alpha\, T_\mu\,, \tag{2.4}$$

where T_μ satisfies the SU(3) Lie algebra

$$[T_\mu, T_\nu] = i f_{\mu\nu\rho} T_\rho\,. \tag{2.5}$$

Hence one obtains a classical solution for each of the N-dimensional representations of the SU(3) Lie algebra. Let us consider the case in which T_μ is given by the irreducible (m, n) representation $T_\mu^{(m,n)}$. Such a solution exists when the size of the matrices A_μ is

$$N = \frac{1}{2}(m+1)(n+1)(m+n+2)\,. \tag{2.6}$$

The explicit form of $T_\mu^{(m,n)}$ is given in the appendix A.

The space represented by the matrices $A_\mu = \alpha\, T_\mu^{(m,n)}$ has SU(3) isometry. There are two kinds of manifold whose isometry is SU(3). One is SU(3)/U(2), and the other is SU(3)/(U(1) × U(1)). In fact the SU(3)/U(2) space corresponds to the $(m, 0)$ or the $(0, n)$ representation, whereas the SU(3)/(U(1) × U(1)) space corresponds to the (m, m) representation.

In what follows we consider the $\mathrm{CP}^2 = \mathrm{SU(3)/U(2)}$ space, which is given by

$$A_\mu^{(\mathrm{CP}^2)} \equiv \alpha\, T_\mu^{(m,0)}\,. \tag{2.7}$$

Note that this solution exists only for

$$N = \frac{1}{2}(m+1)(m+2) = 3, 6, 10, 15, 21, \cdots\ . \tag{2.8}$$

This is in contrast with the fuzzy S^2 case [59], where the corresponding solution exists for arbitrary N. As in ref. [59] we define the "radius-squared matrix"

$$Q = (A_\mu)^2\,, \tag{2.9}$$

which is useful for distinguishing various solutions. Using the identity (A.8), we obtain

$$Q = \rho^2\, \mathbf{1}_N\,, \qquad \rho = \alpha\sqrt{\frac{m(m+3)}{3}} \tag{2.10}$$

for the CP^2 solution, which implies that the fuzzy CP^2 has the radius ρ.

We note that the model also has the fuzzy S^2 type of classical solutions. As an example, let us consider

$$A_\mu^{(S^2)} \equiv \begin{cases} \alpha\, L_\mu^{(N)} & \text{for } \mu = 1,2,3\,, \\ 0 & \text{otherwise}\,, \end{cases} \tag{2.11}$$

where $L_\mu^{(N)}$ is the N-dimensional irreducible representation of the SU(2) Lie algebra. The radius-squared matrix Q is given in this case as

$$Q = R^2\, \mathbf{1}_N\,, \qquad R = \frac{1}{2}\,\alpha\,\sqrt{N^2-1}\,. \tag{2.12}$$

Although there are other fuzzy S^2 solutions with a smaller radius, the one given above is considered the most relevant since it has the smallest action among them.

3. Properties of the fuzzy CP^2

In order to study the properties of the fuzzy CP^2, we simulate the model (2.1) using (2.7) as the initial configuration. We apply the heat bath algorithm developed in ref. [70] for the pure Yang-Mills model by implementing the cubic term as explained in the appendix A of ref. [59].

From perturbative calculations, it turns out natural to fix the rescaled parameter

$$\bar{\alpha} = \alpha\, N^{\frac{1}{4}}\,, \tag{3.1}$$

when we take the large-N limit. In figure 1 we plot the Monte Carlo results for the action $\langle S\rangle$ and the "extent of space-time" $\langle \frac{1}{N}\mathrm{tr}(A_\mu)^2\rangle$ (with an appropriate normalization factor) against the rescaled parameter $\bar{\alpha}$ for $N = 10, 15, 21, 28$ ($m = 3,4,5,6$). We observe a discontinuity around

$$\bar{\alpha} = \bar{\alpha}_{\rm cr}^{(CP^2)} \simeq 2.3\,, \tag{3.2}$$

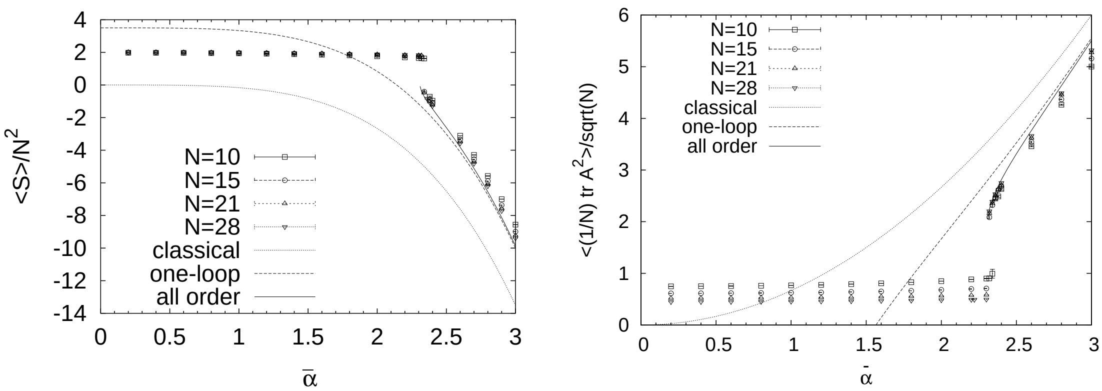

Figure 1: The quantities $\frac{1}{N^2}\langle S\rangle$ and $\frac{1}{\sqrt{N}}\langle \frac{1}{N}\mathrm{tr}(A_\mu)^2\rangle$ are plotted against $\bar{\alpha} = \alpha N^{\frac{1}{4}}$ for $N = 10, 15, 21, 28$ ($m = 3,4,5,6$) for the fuzzy CP^2 start. The dotted, dashed and solid lines represent the classical, one-loop and all order results, respectively, at large N.

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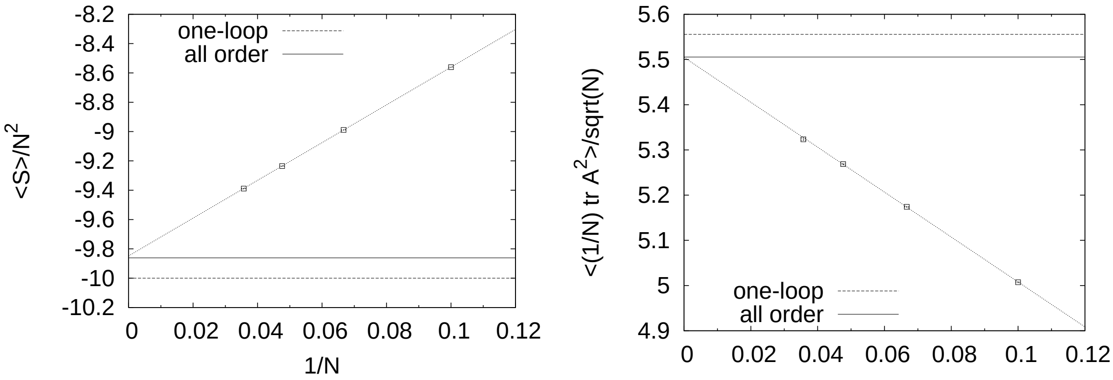

Figure 2: The quantities $\frac{1}{N^2}\langle S\rangle$ (left) and $\frac{1}{\sqrt{N}}\langle \frac{1}{N}\mathrm{tr}(A_\mu)^2\rangle$ (right) are plotted against $\frac{1}{N}$ for $\bar{\alpha} = 3.0$ for the fuzzy CP^2 start. The dotted lines represent a linear fit, and the horizontal solid (dashed) lines represent the all order (one-loop) results obtained analytically at $N = \infty$. The large-N extrapolation demonstrates excellent agreement with the all order result.

which suggests the existence of a first-order phase transition. An analogous first-order phase transition has been found also in the 3d YMCS model [59]. The critical point (3.2) agrees well with the analytical result (C.10) with $k = 1$.

Below the critical point, the Monte Carlo results are almost independent of α. This is because the cubic term in the action (2.1) takes small values, and hence it does not play any role in this regime. Note, in particular, that $\frac{1}{N}\langle \mathrm{tr}(A_\mu)^2\rangle \simeq \mathrm{O}(1)$ as in the pure Yang-Mills model ($\alpha = 0$) [70], which means that $\frac{1}{N}\langle \mathrm{tr}(A_\mu)^2\rangle$ has different large-N behaviors in the two phases, as clearly seen in figure 1.

Above the critical point, we observe that Monte Carlo results approach the all order results (C.40), (C.41) with $k = 1$ as N increases. In order to clarify the finite-N effects, in figure 2 we plot the same quantities against $\frac{1}{N}$ for $N = 10, 15, 21, 28$ ($m = 3, 4, 5, 6$) with fixed $\bar{\alpha} = 3.0 > \bar{\alpha}_{\mathrm{cr}}^{(\mathrm{CP}^2)}$. The data can be nicely fitted to a straight line, which implies that the leading finite-N effect is of $\mathrm{O}(\frac{1}{N})$. This allows us to make a reliable extrapolation to $N = \infty$, and we find excellent agreement with the all order results.

4. Properties of the fuzzy S^2

In this section we study the properties of the fuzzy S^2 by Monte Carlo simulation using (2.11) as the initial configuration. Perturbative calculations suggest that the natural definition of the rescaled parameter in this case is

$$\tilde{\alpha} = \alpha\, N^{\frac{1}{2}}\,, \tag{4.1}$$

unlike (3.1) in the fuzzy CP^2 case. In figure 3 we plot $\frac{1}{N^2}\langle S\rangle$ and $\frac{1}{N}\langle \frac{1}{N}\mathrm{tr}(A_\mu)^2\rangle$ against $\tilde{\alpha}$. We observe a discontinuity at

$$\tilde{\alpha} = \tilde{\alpha}_{\mathrm{cr}}^{(\mathrm{S}^2)} \simeq 3.2\,, \tag{4.2}$$

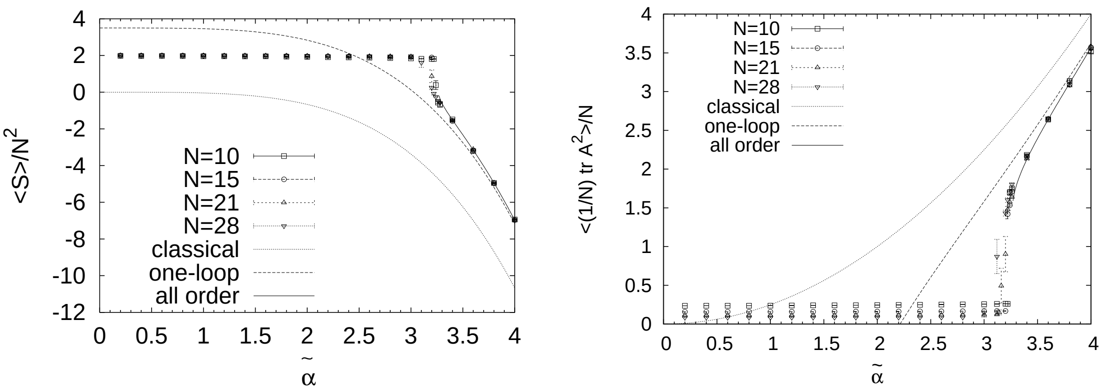

Figure 3: The quantities $\frac{1}{N^2}\langle S\rangle$ and $\frac{1}{N}\langle\frac{1}{N}\mathrm{tr}(A_\mu)^2\rangle$ are plotted against $\tilde{\alpha} = \alpha N^{\frac{1}{2}}$ for $N = 10, 15, 21, 28$ with the fuzzy S^2 start. The dotted, dashed and solid lines represent the classical, one-loop and all order results, respectively, at large N.

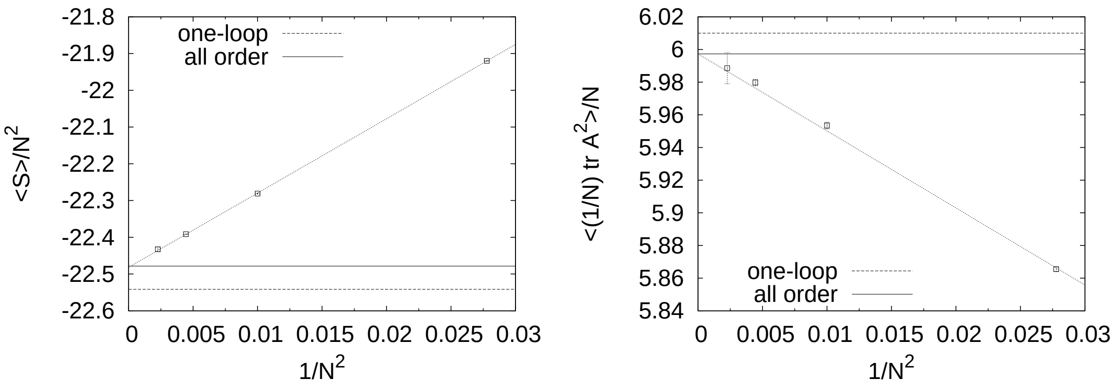

Figure 4: The quantities $\frac{1}{N^2}\langle S\rangle$ (left) and $\frac{1}{N}\langle\frac{1}{N}\mathrm{tr}(A_\mu)^2\rangle$ (right) are plotted against $\frac{1}{N^2}$ for $\tilde{\alpha} = 5.0$ with the fuzzy S^2 start. The dotted lines represent a linear fit, and the horizontal solid (dashed) lines represent the all order (one-loop) results obtained analytically at $N = \infty$. The large-N extrapolation demonstrates excellent agreement with the all order result.

which suggests the existence of a first-order phase transition. The critical point (4.2) agrees well with the analytical result (D.6) with $k = 1$. In terms of the unrescaled parameter α, the critical point for the fuzzy CP^2 and the fuzzy S^2 are $\alpha_{\mathrm{cr}}^{(\mathrm{CP}^2)} \simeq \frac{2.3}{N^{1/4}}$ and $\alpha_{\mathrm{cr}}^{(\mathrm{S}^2)} \simeq \frac{3.2}{N^{1/2}}$, respectively, which means that $\alpha_{\mathrm{cr}}^{(\mathrm{S}^2)} < \alpha_{\mathrm{cr}}^{(\mathrm{CP}^2)}$. Below the critical point, Monte Carlo results are identical to those for the fuzzy CP^2 start presented in the previous section, as expected.

Above the critical point $\tilde{\alpha} > \tilde{\alpha}_{\mathrm{cr}}^{(\mathrm{S}^2)}$, Monte Carlo results are quite close to the all order results (D.13), (D.14) with $k = 1$. In figure 4 we plot the two quantities against $\frac{1}{N^2}$ for $N = 6, 10, 15, 21$ with fixed $\tilde{\alpha} = 5.0 > \tilde{\alpha}_{\mathrm{cr}}^{(\mathrm{S}^2)}$. Monte Carlo results can be nicely fitted to a straight line, which implies that the leading finite-N effect is $\mathrm{O}(\frac{1}{N^2})$, as compared with $\mathrm{O}(\frac{1}{N})$ for the fuzzy CP^2 case. The large-N extrapolation demonstrates perfect agreement with the all order results obtained in the large-N limit.

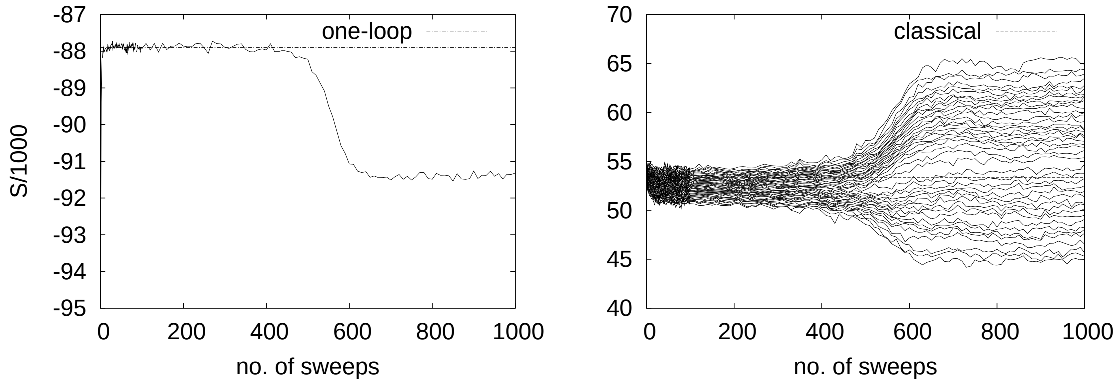

Figure 5: The history of the action S (left) and the eigenvalues of the radius-squared matrix Q (right) for $\alpha = 2.0$ and $N = 42$ using the $k = 2$ coincident fuzzy CP^2 as the initial configuration. The horizontal line in the left (right) plot represents the one-loop (classical) result for the $k = 2$ coincident fuzzy CP^2 at $\alpha = 2.0$ and $N = 42$.

5. CP^2 versus S^2 —which is the true vacuum? —

In the previous two sections, we have seen that both the fuzzy CP^2 and the fuzzy S^2 are stable for sufficiently large α. In this section we discuss which of the two describes the true vacuum. For that purpose we compare the free energy for the fuzzy CP^2 and the fuzzy S^2, which are obtained to all orders in perturbation theory in the large-N limit as (C.37) and (D.12), respectively. Setting $k = 1$ and rewriting in terms of the unrescaled parameter α, the free energy reads

$$W^{(\mathrm{CP}^2)} \simeq N^2\left(-\frac{\alpha^4 N}{6} + 6\log\alpha + \log(8N^7) - \frac{9}{\alpha^4 N} - \frac{63}{\alpha^8 N^2} - \frac{1485}{2\alpha^{12}N^3} - \cdots\right), \quad (5.1)$$

$$W^{(\mathrm{S}^2)} \simeq N^2\left(-\frac{\alpha^4 N^2}{24} + 6\log\alpha + \log N^{10} - \frac{36}{\alpha^4 N^2} - \frac{1008}{\alpha^8 N^4} - \frac{47520}{\alpha^{12}N^6} - \cdots\right). \quad (5.2)$$

Thus in the region $\alpha \geq \alpha_{\mathrm{cr}}^{(\mathrm{CP}^2)} = \frac{2.3}{N^{1/4}}$, where both the fuzzy CP^2 and the fuzzy S^2 exist, we find that the fuzzy S^2 has smaller free energy than the fuzzy CP^2 at large N. Therefore the fuzzy CP^2 cannot be the true vacuum in the present model.

We note, however, that the fuzzy CP^2 appears to be very stable. For instance, we have performed a simulation with the fuzzy CP^2 start for $N = 10$ ($m = 3$) and $\alpha = 1.4$, which is just above the critical point. The fuzzy CP^2 does not decay into the fuzzy S^2 even after 10^7 sweeps of the heat bath algorithm. This suggests the existence of a potential barrier between the two vacua, which presumably increases with N. We therefore consider that the fuzzy CP^2 stabilizes due to the suppression of tunneling effects in the large-N limit.

6. Properties of the k coincident fuzzy CP^2

In this section we discuss the properties of the k coincident fuzzy CP^2 configuration

$$A_\mu^{(k\,\mathrm{CP}^2)} \equiv \alpha\, T_\mu^{(m,0)} \otimes \mathbf{1}_k\,, \quad (6.1)$$

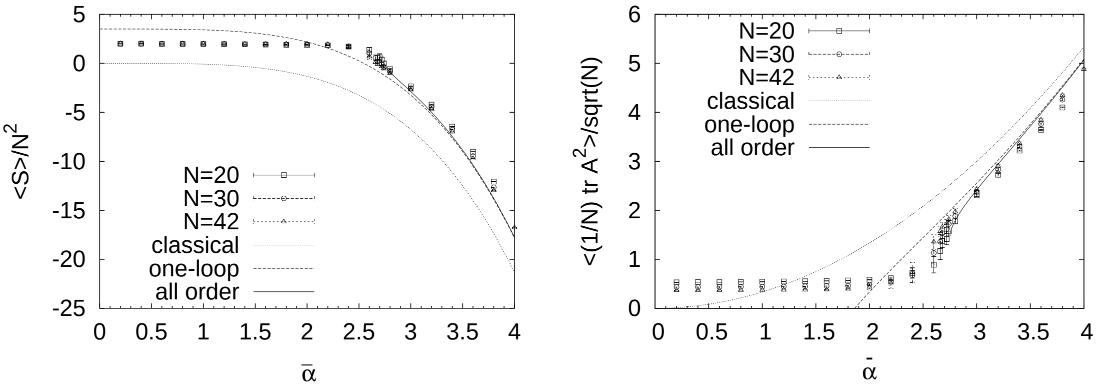

Figure 6: The quantities $\frac{1}{N^2}\langle S\rangle$ and $\frac{1}{\sqrt{N}}\langle \frac{1}{N}\mathrm{tr}(A_\mu)^2\rangle$ are plotted against $\bar{\alpha} = \alpha N^{\frac{1}{4}}$ for $N = 20, 30, 42$ ($m = 3, 4, 5$) with the $k = 2$ coincident fuzzy CP^2 start. The dotted, dashed and solid lines represent the classical, one-loop and all order results, respectively, at large N.

which is also a classical solution of the model. The size of the matrices should now be

$$N = \frac{1}{2}\, k\, (m+1)(m+2)\,. \tag{6.2}$$

Such a configuration is important since it gives rise to a gauge theory on the fuzzy CP^2 with the gauge group of rank k.

We have performed Monte Carlo simulation with the initial configuration given by (6.1) with $k = 2$. In figure 5 we plot the action S and the eigenvalues of the radius-squared matrix Q defined by (2.9) against the number of "sweeps" in the heat bath algorithm [70]. We observe that the $k = 2$ fuzzy CP^2 decays after 600 sweeps.

Although the $k = 2$ coincident fuzzy CP^2 is thus only meta-stable, we may measure various observables before it actually decays in Monte Carlo simulation. In figure 6 we plot the results against $\bar{\alpha}$ for $N = 20, 30, 42$ ($m = 3, 4, 5$). We observe a discontinuity at

$$\bar{\alpha} = \bar{\alpha}_{\mathrm{cr}}^{(k=2\,\mathrm{CP}^2)} \simeq 2.7\,, \tag{6.3}$$

which agrees with the analytical result (C.10). Above the critical point, our Monte Carlo results agree well with the all order results (C.40), (C.41) with $k = 2$.

7. Dynamical gauge group

In the previous section, we have seen in Monte Carlo simulation that the k coincident fuzzy CP^2 is unstable in the $k = 2$ case. This instability is related to the zero modes that appear in the perturbation theory around these configurations. (See appendix C.1.) Although we cannot exclude the possibility that the instability disappears in the large-N limit, we show that the multi-fuzzy CP^2 cannot be the true vacuum anyway, by comparing the free energy calculated omitting the zero modes. The explicit form of the free energy to all orders in perturbation theory is obtained as (C.37) above the critical point (C.10), which we plot for $k = 1, \cdots, 6$ in figure 7 (left). We find that the $k = 1$ case gives the smallest free energy

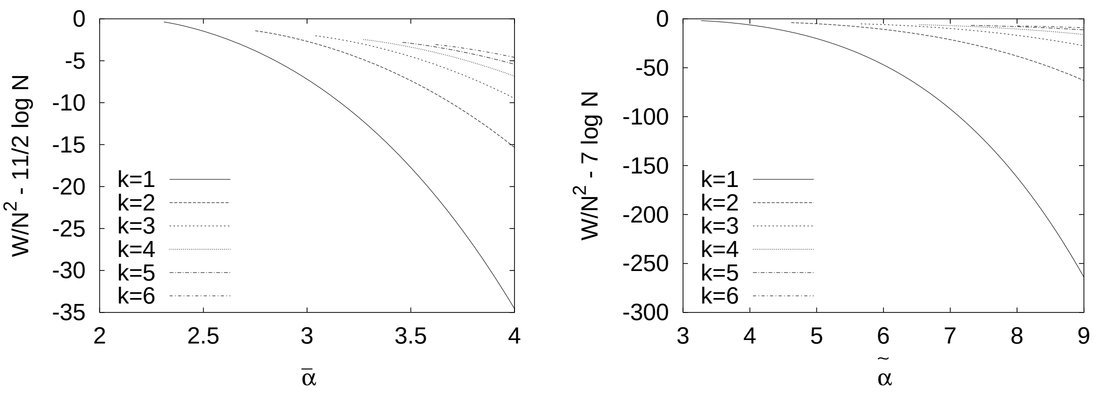

Figure 7: The free energy obtained to all orders in perturbation theory is plotted above the critical point for the k coincident fuzzy CP^2 (left) and fuzzy S^2 (right) with $k = 1, \cdots, 6$. We have taken the large-N limit after subtracting the irrelevant constant term proportional to $\log N$. In both cases, the $k = 1$ case represented by the solid line gives the smallest free energy.

for all values of $\bar{\alpha}$. Thus, we conclude that the dynamical gauge group for the fuzzy CP^2 is of rank one.

We repeat the same analysis for the fuzzy S^2 case. The free energy for the k coincident fuzzy S^2 is given to all orders in perturbation theory above the critical point (D.6) by (D.12), which we plot for $k = 1, \cdots, 6$ in figure 7 (right). We find that the $k = 1$ case gives the smallest free energy for all values of $\tilde{\alpha}$. Therefore, the dynamical gauge group is of rank one in this case as well.

Let us emphasize that the existence of the first-order phase transition plays an important role in arriving at these conclusions. As one can see from the free energy (C.37), (D.12), the classical term favors small k, while the one-loop term proportional to $\log k$ favors large k. Therefore, if we disregard the existence of the first-order phase transition, the rank of the dynamical gauge group could be $k > 1$ at small α. What actually happens is that the critical point (C.10), (D.6) increases with k, and since we have to restrict ourselves to the region above the critical point, the one-loop term cannot compensate the effect of the classical term.

8. Summary and discussions

In this paper we have applied nonperturbative techniques and ideas, which have been developed in ref. [59], to a four-dimensional fuzzy manifold, the fuzzy CP^2. The present model may be considered as a natural extension of our previous model in the sense that the epsilon tensor, which is nothing but the structure constant of SU(2), is now replaced by that of SU(3). Since the number of bosonic matrices should be equal to or larger than the dimensionality of the algebra, it is taken to be 8 instead of 3. Unlike the fuzzy S^4 in the matrix model with a quintic term [60], the fuzzy CP^2 in the present model is nonperturbatively stable in the large-N limit despite the fact that it has larger free

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energy than the fuzzy S^2 in the same model. Thus the model provides a nonperturbative definition of a gauge theory on the fuzzy CP^2. It would be interesting to investigate the field theoretical aspects of this model as in the case of noncommutative torus [41].

From the viewpoint of the dynamical generation of space-time, however, we should note that the fuzzy CP^2 cannot be realized as the *true vacuum* since it has larger free energy than the fuzzy S^2. This conclusion is in contrast to the results [53 − 55, 57] obtained in the IIB matrix model [43], where four-dimensional space-time is shown to have smaller free energy than the space-time with other dimensionality. We should also note that the gauge group dynamically generated for the cases studied in this paper as well as in the previous work [59] turns out to be of rank one, although there is no reason for it *a priori*. Indeed ref. [78] presents an explicit model in which the gauge group with higher rank is realized in the true vacuum. In the IIB matrix model, we feel that supersymmetry plays an important role in obtaining four-dimensional space-time as well as a gauge group of sufficiently high rank that can accommodate the Standard Model. We would like to report on these issues in the near future.

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A. Explicit form of the CP^2 configuration

In this section we present the explicit form of the representation matrix $T_\mu^{(m,n)}$ of the SU(3) algebra. This, in particular, provides us with the explicit form of the fuzzy CP^2 configuration $A_\mu^{(CP^2)}$ in eq. (2.7).

For that purpose we need to introduce the so-called (anti-)symmetric tensor product. Let us denote the matrix element of the matrix A for the orthonormal states $|i\rangle$ and $|j\rangle$ as $(A)_{ij} = \langle i|A|j\rangle$. The usual tensor product is defined by

$$\langle i_1, i_2|A \otimes B|j_1, j_2\rangle = \langle i_1|A|j_1\rangle\langle i_2|B|j_2\rangle \,, \tag{A.1}$$

where $|j_1, j_2\rangle = |j_1\rangle|j_2\rangle$. The (anti-)symmetric tensor product are defined through their matrix elements

$$_{\rm sym}\langle i_1, i_2|A \otimes B|j_1 j_2\rangle_{\rm sym} \,, \tag{A.2}$$

$$_{\rm asym}\langle i_1, i_2|A \otimes B|j_1 j_2\rangle_{\rm asym} \,, \tag{A.3}$$

where $|j_1, j_2\rangle_{\rm sym}$ and $|j_1, j_2\rangle_{\rm asym}$ are the orthonormal (anti-)symmetrized state defined by

$$|j_1, j_2\rangle_{\rm sym} = \begin{cases} |j_1\rangle|j_2\rangle & \text{for } j_1 = j_2\,, \\ \frac{1}{\sqrt{2}}(|j_1\rangle|j_2\rangle + |j_2\rangle|j_1\rangle) & \text{for } j_1 \neq j_2\,, \end{cases} \tag{A.4}$$

$$|j_1, j_2\rangle_{\rm asym} = \frac{1}{\sqrt{2}}(|j_1\rangle|j_2\rangle - |j_2\rangle|j_1\rangle) \qquad \text{for } j_1 \neq j_2\,, \tag{A.5}$$

respectively. The size of the matrices representing the symmetric tensor product $(A \otimes B)_{\rm sym}$ and the anti-symmetric tensor product $(A \otimes B)_{\rm asym}$ is ${}_{k+1}\mathrm{C}_2$ and ${}_k\mathrm{C}_2$, respectively, where k is the size of the matrices A and B. The (anti-)symmetric tensor product can be generalized straightforwardly to a product of more than two matrices. Then the representation matrix of the $(m, 0)$ representation is given as

$$T_\mu^{(m,0)} = \sum_{j=1}^{m} \Big(\underbrace{\mathbf{1}_3 \otimes \cdots \otimes \mathbf{1}_3}_{j-1} \otimes t_\mu \otimes \underbrace{\mathbf{1}_3 \otimes \cdots \otimes \mathbf{1}_3}_{m-j}\Big)_{\rm sym}\,, \tag{A.6}$$

where t_μ denotes the fundamental $(1, 0)$ representation of the SU(3) Lie algebra, which is given explicitly as

$$t_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad t_2 = \frac{1}{2}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad t_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad t_4 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$t_5 = \frac{1}{2}\begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad t_6 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad t_7 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad t_8 = \frac{1}{2\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

The $(0, n)$ representation can be obtained by simply replacing the fundamental representation t_μ by the anti-fundamental representation, $s_\mu = -t_\mu^*$, in the above definition.

In order to obtain the (m, n) representation, we have to define an orthonormal state $|j_1, \cdots, j_m; k_1, \cdots, k_n\rangle_{\rm mix}$ with a mixed symmetry such that it is symmetric with respect to the first m indices and the last n indices, separately, and anti-symmetric with respect to the exchange of one of the first m indices and one of the last n indices. (This symmetry is exactly the symmetry of the Young tableaux for the (m, n) representation.) We denote the tensor product defined with these states as $(A_1 \otimes \cdots \otimes A_m \otimes B_1 \otimes \cdots \otimes B_n)_{\rm mix}$. Then the representation matrix of the (m, n) representation is given as

$$\begin{aligned} T_\mu^{(m,n)} = \Big(&\sum_{j=1}^{m} \underbrace{\mathbf{1}_3 \otimes \cdots \otimes \mathbf{1}_3}_{j-1} \otimes t_\mu \otimes \underbrace{\mathbf{1}_3 \otimes \cdots \otimes \mathbf{1}_3}_{m-j} \otimes \underbrace{\mathbf{1}_3 \otimes \cdots \otimes \mathbf{1}_3}_{n} \\ &+ \sum_{k=1}^{n} \underbrace{\mathbf{1}_3 \otimes \cdots \otimes \mathbf{1}_3}_{m} \otimes \underbrace{\mathbf{1}_3 \otimes \cdots \otimes \mathbf{1}_3}_{k-1} \otimes s_\mu \otimes \underbrace{\mathbf{1}_3 \otimes \cdots \otimes \mathbf{1}_3}_{n-k}\Big)_{\rm mix}\,. \end{aligned} \tag{A.7}$$

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Note also that

$$
\begin{aligned}
\left(T_\mu^{(m,n)}\right)^2 &= \Bigg(\underbrace{((t_\mu)^2 \otimes \mathbf{1}_3 \otimes \cdots \otimes \mathbf{1}_3) + \cdots}_{m \text{ terms}} + \underbrace{(\mathbf{1}_3 \otimes \cdots \otimes \mathbf{1}_3 \otimes (s_\mu)^2) + \cdots}_{n \text{ terms}} \\
&\quad + \underbrace{(t_\mu \otimes t_\mu \otimes \mathbf{1}_3 \otimes \cdots \otimes \mathbf{1}_3) + \cdots}_{m(m-1) \text{ terms}} + \underbrace{(\mathbf{1}_3 \otimes \cdots \otimes \mathbf{1}_3 \otimes s_\mu \otimes s_\mu) + \cdots}_{n(n-1) \text{ terms}} \\
&\quad + \underbrace{(t_\mu \otimes \cdots \otimes \mathbf{1}_3 \otimes s_\mu \otimes \cdots \otimes \mathbf{1}_3) + \cdots}_{2mn \text{ terms}} \Bigg)_{\text{mix}} \\
&= \left(\frac{4}{3}(m+n) + \frac{1}{3}(m(m-1) + n(n-1)) + \frac{1}{6} \times 2mn\right) \mathbf{1}_N \\
&= \frac{m(m+3) + n(n+3) + mn}{3} \mathbf{1}_N \,, \qquad \text{(A.8)}
\end{aligned}
$$

where we have used the formulae

$$
\begin{aligned}
(t_\mu \otimes t_\mu)_{\text{sym}} &= \frac{1}{3} (\mathbf{1}_3 \otimes \mathbf{1}_3)_{\text{sym}} \,, && \text{(A.9)} \\
(t_\mu \otimes s_\mu)_{\text{asym}} &= \frac{1}{6} (\mathbf{1}_3 \otimes \mathbf{1}_3)_{\text{asym}} \,, && \text{(A.10)} \\
(t_\mu)^2 &= (s_\mu)^2 = \frac{4}{3} \mathbf{1}_3 \,. && \text{(A.11)}
\end{aligned}
$$

B. Perturbative expansion around fuzzy manifolds

In this section we formulate the perturbation theory around the classical solution X_μ given either by (6.1) representing k coincident fuzzy CP^2 or by (D.1) representing k coincident fuzzy S^2. The single fuzzy CP^2 and the single fuzzy S^2 are included as a special case $k = 1$.

Let us evaluate the partition function $Z = \int dA\, e^{-S}$ around the classical solution $A_\mu = X_\mu$ at the one-loop level. We define the measure of the path integral as

$$
dA = \prod_{\mu=1}^{8} \prod_{a=1}^{N^2-1} dA_\mu^a \,, \qquad \text{(B.1)}
$$

where $A_\mu = \sum_{a=1}^{N^2-1} A_\mu^a\, t^a$ with t^a being the generators of $\mathrm{SU}(N)$ normalized as $\mathrm{tr}(t^a t^b) = \delta_{ab}$. We need to fix the gauge since there are flat directions corresponding to the transformation

$$
A_\mu \to A_\mu^g = g\, A_\mu\, g^\dagger \,, \qquad \text{(B.2)}
$$

where g is an element of the coset space $H = \mathrm{U}(N)/\mathrm{U}(k)$. In order to remove the associated zero modes, we introduce the gauge fixing term and the corresponding ghost term

$$
\begin{aligned}
S_{\text{g.f.}} &= -\frac{N}{2} \,\mathrm{tr}\, [X_\mu, A_\mu]^2 \,, && \text{(B.3)} \\
S_{\text{ghost}} &= -N \,\mathrm{tr}\Big([X_\mu, \bar{c}][A_\mu, c] \Big) \,, && \text{(B.4)}
\end{aligned}
$$

where c and $\bar{c}$ are the ghost and anti-ghost fields, respectively. These ghost fields take values in the tangent space of H. We perform the integration over A_μ by decomposing it into the classical background and the fluctuation as $A_\mu = X_\mu + \tilde{A}_\mu$. The partition function can be rewritten as

$$Z = \mathrm{vol}(H)\,\mathcal{N} \int d\tilde{A}\, dc\, d\bar{c}\, e^{-S_{\mathrm{total}}}\,, \tag{B.5}$$

where the total action is defined by

$$S_{\mathrm{total}} = S + S_{\mathrm{g.f.}} + S_{\mathrm{ghost}}\,, \tag{B.6}$$

and it is given explicitly as

$$S_{\mathrm{total}} = S[X] + S_{\mathrm{kin}} + S_{\mathrm{int}}\,, \tag{B.7}$$

$$S_{\mathrm{kin}} = \frac{1}{2} N \,\mathrm{tr}\left(\tilde{A}_\mu [X_\lambda, [X_\lambda, \tilde{A}_\mu]]\right) + N \,\mathrm{tr}\left(\bar{c}\,[X_\lambda, [X_\lambda, c]]\right), \tag{B.8}$$

$$\begin{aligned} S_{\mathrm{int}} = &-N \,\mathrm{tr}\left([\tilde{A}_\mu, \tilde{A}_\nu][X_\mu, \tilde{A}_\nu]\right) - \frac{1}{4} N \,\mathrm{tr}\left([\tilde{A}_\mu, \tilde{A}_\nu]^2\right) \\ &+ \frac{2}{3}\, i\,\alpha f_{\mu\nu\rho}\, N \,\mathrm{tr}\left(\tilde{A}_\mu \tilde{A}_\nu \tilde{A}_\rho\right) + N \,\mathrm{tr}\left(\bar{c}\,[X_\mu, [\tilde{A}_\mu, c]]\right). \end{aligned} \tag{B.9}$$

The normalization factor $\mathcal{N} = (2\pi N)^{-(N^2-k^2)/2}$ in (B.5) can be obtained by following the usual gauge fixing procedure as in ref. [78]. The linear terms in $\tilde{A}_\mu$ cancel since X_μ is assumed to satisfy the classical equation of motion. Since the classical solution X_μ we are considering is proportional to α, we can rescale the matrices as $A_\mu \mapsto \alpha\, A_\mu$, $c \mapsto \alpha\, c$, $\bar{c} \mapsto \alpha\, \bar{c}$, so that all the terms in the total action S_{total} become proportional to α^4. This means that the expansion parameter of the present perturbation theory is $\frac{1}{\alpha^4}$. The volume of the coset space H in (B.5), $\mathrm{vol}(H) = \mathrm{vol}(\mathrm{U}(N))/\mathrm{vol}(\mathrm{U}(k))$, can be obtained by using the formula

$$\mathrm{vol}(\mathrm{U}(p)) = \frac{(2\pi)^{p(p+1)/2}}{(p-1)!\,(p-2)!\cdots 1!\,0!}\,. \tag{B.10}$$

We calculate the free energy $W = -\log Z$ as a perturbative expansion $W = \sum_{j=0}^{\infty} W_j$, where $W_j = \mathrm{O}(\alpha^{4(1-j)})$ comes from the j-loop contribution. The classical part is obtained as $W_0 = S[X]$, which is nothing but the action evaluated at the classical solution $A_\mu = X_\mu$.

Introducing the operator $\mathcal{P}_\mu$

$$\mathcal{P}_\mu M \stackrel{\mathrm{def}}{=} [X_\mu, M]\,, \tag{B.11}$$

which acts on a $N \times N$ traceless hermitian matrix M, the kinetic term (B.8) reads

$$S_{\mathrm{kin}} = N \,\mathrm{tr}\left\{\frac{1}{2}\tilde{A}_\mu (\mathcal{P}_\lambda)^2 \tilde{A}_\mu + \bar{c}(\mathcal{P}_\lambda)^2 c\right\}. \tag{B.12}$$

The one-loop term can therefore be obtained as

$$W_1 = 3\,\mathcal{T}r \log\{N(\mathcal{P}_\lambda)^2\} - \log\{\mathrm{vol}(H)\mathcal{N}\}\,, \tag{B.13}$$

where the symbol $\mathcal{T}r$ denotes the trace in the space of traceless hermitian matrices.

C. Perturbative calculations for the fuzzy CP^2

In this section we focus on the case where the classical solution X_μ is taken to be the k coincident fuzzy CP^2 (6.1). The results for the single fuzzy CP^2 can be readily obtained by setting $k = 1$.

C.1 One-loop calculation of free energy

Let us calculate the free energy up to one-loop. The classical part is obtained as

$$W_0 = -\frac{1}{6} N^2 \alpha^4 (n-1)\,, \tag{C.1}$$

where we have defined

$$n \equiv \frac{N}{k} = \frac{1}{2}(m+1)(m+2)\,, \tag{C.2}$$

and used the relation

$$f_{\mu\nu\rho}\, f_{\mu\nu\rho'} = 3\,\delta_{\rho\rho'}\,. \tag{C.3}$$

Next we evaluate the one-loop contribution W_1 in (B.13). In order to solve the eigenvalue problem of the operator $(\mathcal{P}_\lambda)^2$, we introduce an analog of matrix spherical harmonics in the fuzzy S^2 case [59]. Let us denote it as $\{Y_{st}\}$, where the indices s and t run over $s = 0, 1, \cdots, m$ and $t = 1, \cdots, (s+1)^3$, respectively. It gives a complete basis for the space of $n \times n$ matrices. (Note, for instance, $\sum_{s=0}^{m} (s+1)^3 = n^2$.) For a given s, Y_{st} transforms as a (s,s)-type irreducible representation of SU(3) under the adjoint operation $[T_\mu^{(m,0)}, \ \cdot\ ]$. For more details of Y_{st}, see refs. [28, 62−65]. We also introduce $k \times k$ matrices $\mathbf{e}^{(a,b)}$, whose (a,b) element is 1 and all the other elements are zero. Then, as a complete basis of $N \times N$ matrices, we define

$$\mathcal{Y}_{st}^{(a,b)} \equiv Y_{st} \otimes \mathbf{e}^{(a,b)}\,, \tag{C.4}$$

which satisfies the relation

$$\mathrm{tr}\left(\mathcal{Y}_{st}^{(a,b)\dagger} \mathcal{Y}_{s't'}^{(a',b')}\right) = \delta_{ss'}\delta_{tt'}\delta_{aa'}\delta_{bb'}\,. \tag{C.5}$$

The eigenvalue problem of the operator $(\mathcal{P}_\lambda)^2$ can be solved as

$$(\mathcal{P}_\lambda)^2 \mathcal{Y}_{st}^{(a,b)} = \alpha^2 s\,(s+2) \mathcal{Y}_{st}^{(a,b)}\,. \tag{C.6}$$

Note that $\mathcal{Y}_{00}^{(a,b)}$ for all the (a,b) blocks are the zero modes. In the $k=1$ case, the zero mode should be excluded since A_μ are traceless. For $k \geq 2$ the tracelessness condition removes only one of the k^2 zero modes. Here we omit the rest of them by hand [1]. Then the one-loop contribution W_1 is given by

$$W_1 = 3k^2 \sum_{s=1}^{m} (s+1)^3 \log\left[N\alpha^2 s(s+2)\right] - \log\{\mathrm{vol}(H)\mathcal{N}\}\,. \tag{C.7}$$

[1] In fact these zero modes are responsible for the instability of the multi-fuzzy CP^2 discussed in section 6. (See appendix D of ref. [59] for more in-depth discussions on this point in an analogous model.) However, the number of zero modes, $k^2 - 1$, is negligible compared with the dimension of the configuration space, which is of $\mathrm{O}(N^2)$. Indeed figure 6 shows that the results obtained by omitting the zero modes are in reasonable agreement with Monte Carlo results obtained before the decay of the multi-fuzzy CP^2 actually takes place.

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The one-loop free energy is obtained at large N as

$$W_0 + W_1 \simeq N^2 \left(-\frac{\bar{\alpha}^4}{6k} + 6 \log \bar{\alpha} + \log \frac{8N^{\frac{11}{2}}}{k^3} - \frac{9}{4} \right) , \tag{C.8}$$

where the rescaled parameter $\bar{\alpha}$ is defined by (3.1).

C.2 Derivation of the critical point

In section 3 we observed a phase transition in Monte Carlo simulation starting from the fuzzy CP^2 configuration. We can derive the critical point in the same way as in the 3d YMCS model [59]. Let us consider the effective action for a one-parameter family of configurations $A_\mu = \beta T_\mu^{(m,0)} \otimes \mathbf{1}_k$. At the one-loop level, it is obtained at large N as

$$\Gamma_{1-\text{loop}}(\bar{\beta}) \simeq N^2 \left\{ \frac{2}{3k} \left(\frac{3\bar{\beta}^4}{4} - \bar{\alpha}\bar{\beta}^3 \right) + 6 \log \bar{\beta} + \log \frac{8N^{\frac{11}{2}}}{k^3} - \frac{9}{4} \right\} , \tag{C.9}$$

where $\bar{\beta} = \beta N^{\frac{1}{4}}$. The function of $\bar{\beta}$ on the right hand side has a local minimum for

$$\bar{\alpha} > \bar{\alpha}_{\text{cr}}^{(k\mathrm{CP}^2)} = \frac{4}{\sqrt{3}} k^{\frac{1}{4}} = 2.3094011 \cdots \times k^{\frac{1}{4}} , \tag{C.10}$$

which determines the (lower) critical point of the first-order phase transition. In fact it is known that the effective action is saturated at one loop in the large-N limit in the case of fuzzy S^2 or fuzzy $\mathrm{S}^2 \times \mathrm{S}^2$ [32, 57]. This is the case also for the fuzzy CP^2. Therefore, the critical point obtained above is expected to be correct to all orders in perturbation theory.

C.3 One-loop calculation of various observables

In this section we extend the perturbative calculation to various observables including those which are studied by Monte Carlo simulation in section 3 and 6. The zero modes, which appear for $k \geq 2$, is omitted as in the evaluation of the free energy given in appendix C.1. We note that the number of loops in the relevant diagrams can be less than the order of $\frac{1}{\alpha^4}$ in the perturbative expansion since we are expanding the theory around a nontrivial background. At the one-loop level, the only nontrivial task is to evaluate the tadpole $\langle (\tilde{A}_\mu)_{ij} \rangle$ explicitly.

C.3.1 Propagators and the tadpole

The propagators for $\tilde{A}_\mu$ and the ghosts are given as

$$\left\langle (\tilde{A}_\mu)_{ij} (\tilde{A}_\nu)_{kl} \right\rangle_0 = \delta_{\mu\nu} \sum_{ab} \sum_{s=1}^{m} \sum_{t=1}^{(s+1)^3} \frac{1}{N\alpha^2 s(s+2)} \left(\mathcal{Y}_{st}^{(a,b)} \right)_{ij} \left(\mathcal{Y}_{st}^{(a,b)\dagger} \right)_{kl} , \tag{C.11}$$

$$\left\langle (c)_{ij} (\bar{c})_{kl} \right\rangle_0 = \sum_{ab} \sum_{s=1}^{m} \sum_{t=1}^{(s+1)^3} \frac{1}{N\alpha^2 s(s+2)} \left(\mathcal{Y}_{st}^{(a,b)} \right)_{ij} \left(\mathcal{Y}_{st}^{(a,b)\dagger} \right)_{kl} , \tag{C.12}$$

where the symbol $\langle \cdot \rangle_0$ refers to the expectation value calculated using the kinetic term S_{kin} in (B.8) only.

Due to the symmetries, the tadpole $\langle \tilde{A}_\mu \rangle$ can be expressed as

$$\langle \tilde{A}_\mu \rangle = c\, X_\mu \tag{C.13}$$

with some coefficient c. Using the identity

$$\begin{aligned} \mathrm{tr}\left(X_\mu \langle \tilde{A}_\mu \rangle \right) &= c\, \mathrm{tr}(X_\mu X_\mu) \\ &= \frac{cN}{3} \alpha^2 m(m+3)\,, \end{aligned} \tag{C.14}$$

the coefficient c can be determined by calculating the left hand side of (C.14).

At the leading order in $\frac{1}{\alpha^4}$, we have

$$\begin{aligned} \frac{1}{N} \mathrm{tr}\left(X_\mu \langle \tilde{A}_\mu \rangle_{1-\mathrm{loop}} \right) &= \left\langle \mathrm{tr}(X_\mu \tilde{A}_\mu)\, \mathrm{tr}\left([\tilde{A}_\nu, \tilde{A}_\rho][X_\nu, \tilde{A}_\rho] \right) \right\rangle_0 \\ &\quad - \left\langle \mathrm{tr}(X_\mu \tilde{A}_\mu)\, \mathrm{tr}\left(\frac{2}{3}\, i\, \alpha f_{\nu\rho\sigma} \tilde{A}_\nu \tilde{A}_\rho \tilde{A}_\sigma \right) \right\rangle_0 \\ &\quad - \left\langle \mathrm{tr}(X_\mu \tilde{A}_\mu)\, \mathrm{tr}\left(\bar{c}\, [X_\nu, [\tilde{A}_\nu, c]] \right) \right\rangle_0 . \end{aligned} \tag{C.15}$$

Using the fact that X_μ is a linear combination of $(Y_{s=1,t} \otimes \mathbf{1}_k)$, we can calculate (C.15) similarly to the previous section. After some algebra we arrive at

$$\mathrm{tr}\left(X_\mu \langle \tilde{A}_\mu \rangle_{1-\mathrm{loop}} \right) = -\frac{2k^2}{N\alpha^2}(n^2-1)\,. \tag{C.16}$$

Using (C.14) we obtain

$$\langle \tilde{A}_\mu \rangle_{1-\mathrm{loop}} = -\frac{6}{\alpha^4} \frac{n^2-1}{n^2 m(m+3)}\, X_\mu \simeq -\frac{3}{n\alpha^4}\, X_\mu\,. \tag{C.17}$$

C.3.2 One-loop results for various observables

Using the propagator and the tadpole obtained in the previous section, we can evaluate various observables easily at the one-loop level.

The "extent of space-time" $\langle \frac{1}{N} \mathrm{tr}(A_\mu)^2 \rangle$ can be evaluated as

$$\begin{aligned} \left\langle \frac{1}{N} \mathrm{tr}(A_\mu)^2 \right\rangle_{1-\mathrm{loop}} &= \frac{1}{N} \left[\mathrm{tr}(X_\mu X_\mu) + 2\, \mathrm{tr}\left(X_\mu \langle \tilde{A}_\mu \rangle_{1-\mathrm{loop}} \right) + \langle \mathrm{tr}(\tilde{A}_\mu)^2 \rangle_0 \right] \\ &= \alpha^2 \left[\frac{1}{3} m(m+3) - \frac{4}{\alpha^4} \frac{n^2-1}{n^2} + \frac{8}{n^2 \alpha^4} \sum_{s=1}^{m} \frac{(s+1)^3}{s(s+2)} \right] . \end{aligned} \tag{C.18}$$

At large N with fixed $\bar{\alpha}$, we obtain

$$\frac{1}{\sqrt{N}} \left\langle \frac{1}{N} \mathrm{tr}(A_\mu)^2 \right\rangle_{1-\mathrm{loop}} \simeq \frac{2\bar{\alpha}^2}{3k} - \frac{4}{\bar{\alpha}^2}\,. \tag{C.19}$$

The expectation value of the Chern-Simons term

$$M = \frac{2\, i}{3N} f_{\mu\nu\rho} \mathrm{tr}(A_\mu A_\nu A_\rho) \tag{C.20}$$

can be evaluated as

$$\begin{aligned}\langle M\rangle_{1-\text{loop}} &= \frac{2i}{3N} f_{\mu\nu\rho}\left[\text{tr}(X_\mu X_\nu X_\rho) + 3\,\text{tr}\left(X_\mu X_\nu \langle\tilde{A}_\rho\rangle_{1-\text{loop}}\right)\right] \\ &= -\frac{\alpha^3}{3} m(m+3) + \frac{6(n^2-1)}{\alpha n^2}\,. \end{aligned} \tag{C.21}$$

At large N with fixed $\bar{\alpha}$, we get

$$\frac{1}{N^{\frac{1}{4}}}\langle M\rangle_{1-\text{loop}} \simeq -\frac{2\bar{\alpha}^3}{3\,k^2} + \frac{6}{\bar{\alpha}}\,. \tag{C.22}$$

The observable $\langle \frac{1}{N}\text{tr}(F_{\mu\nu})^2\rangle$ can be calculated in a similar manner, but it is easier to make use of the Schwinger-Dyson equation

$$\left\langle \frac{1}{N}\text{tr}(F_{\mu\nu})^2 + 3\alpha M\right\rangle = 8\left(1-\frac{1}{N^2}\right)\,, \tag{C.23}$$

from which we obtain

$$\left\langle \frac{1}{N}\text{tr}(F_{\mu\nu})^2\right\rangle_{1-\text{loop}} = 8\left(1-\frac{1}{N^2}\right) - 3\alpha\langle M\rangle_{1-\text{loop}} \simeq \frac{2\bar{\alpha}^4}{k} - 10\,. \tag{C.24}$$

Combining (C.21) and (C.24), we get

$$\frac{1}{N^2}\langle S\rangle_{1-\text{loop}} = \frac{1}{4}\left\langle \frac{1}{N}\text{tr}(F_{\mu\nu})^2\right\rangle_{1-\text{loop}} + \alpha\,\langle M\rangle_{1-\text{loop}} \simeq -\frac{\bar{\alpha}^4}{6\,k} + \frac{7}{2}\,. \tag{C.25}$$

C.3.3 An alternative derivation

Since $\text{tr}(F_{\mu\nu})^2$ and M are the quantities that appear in the action S, we can obtain their expectation values easily by using the free energy (C.8) calculated for the k coincident fuzzy CP^2. Let us consider the action

$$S(\beta_1,\beta_2;\alpha) = N\text{tr}\left(-\frac{\beta_1}{4}[A_\mu,A_\nu]^2\right) + \beta_2 N^2\alpha M\,, \tag{C.26}$$

where we have introduced two free parameters β_1 and β_2, and define the corresponding free energy by

$$\mathrm{e}^{-W(\beta_1,\beta_2;\alpha)} = \int dA\,\mathrm{e}^{-S(\beta_1,\beta_2;\alpha)}\,. \tag{C.27}$$

By rescaling the integration variables as $A_\mu \mapsto \beta_1^{-\frac{1}{4}}A_\mu$, we find

$$W(\beta_1,\beta_2;\alpha) = 2(N^2-1)\log\beta_1 + W(1,1;\alpha\beta_2\beta_1^{-\frac{3}{4}})\,. \tag{C.28}$$

Then $\langle\text{tr}(F_{\mu\nu})^2\rangle$, $\langle M\rangle$ and $\langle S\rangle$ can be obtained by

$$\left\langle \frac{1}{N}\text{tr}(F_{\mu\nu})^2\right\rangle = \frac{4}{N^2}\left.\frac{\partial W(\beta_1,\beta_2;\alpha)}{\partial\beta_1}\right|_{\beta_1=\beta_2=1} = 8\left(1-\frac{1}{N^2}\right) - \frac{3\bar{\alpha}}{N^2}\frac{\partial W(1,1;\alpha)}{\partial\bar{\alpha}}\,, \tag{C.29}$$

$$\langle M\rangle = \frac{1}{\alpha N^2}\left.\frac{\partial W(\beta_1,\beta_2;\alpha)}{\partial\beta_2}\right|_{\beta_1=\beta_2=1} = \frac{1}{N^{\frac{7}{4}}}\frac{\partial W(1,1;\alpha)}{\partial\bar{\alpha}}\,, \tag{C.30}$$

$$\frac{1}{N^2}\langle S\rangle = \frac{1}{4}\left\langle\frac{1}{N}\text{tr}(F_{\mu\nu})^2\right\rangle + \alpha\langle M\rangle = 2\left(1-\frac{1}{N^2}\right) + \frac{\bar{\alpha}}{4N^2}\frac{\partial W(1,1;\alpha)}{\partial\bar{\alpha}}\,. \tag{C.31}$$

Using the one-loop result

$$W(1,1;\alpha)_{1-\text{loop}} = -\frac{1}{6}N^2\alpha^4(n-1) + 3k^2\sum_{s=1}^{m}(s+1)^3\log\left[N\alpha^2 s(s+2)\right] - \log\{\text{vol}(H)\mathcal{N}\}\,, \tag{C.32}$$

which follows from (C.1) and (C.7), we can reproduce (C.21) and (C.24).

C.4 All order results from one-loop calculation

Taking advantage of the one-loop saturation of the effective action mentioned at the end of appendix C.2, we can obtain all order results for various quantities in the large-N limit by simply shifting the center of expansion in the one-loop calculation [37]. Similar calculations have been done previously in the 3d YMCS model [66].

Since the free energy and the effective action are related to each other by the Legendre transformation, we can obtain the free energy by evaluating the effective action at its extremum. We consider the expansion around a configuration $A_\mu = \beta\, T_\mu^{(m,0)} \otimes \mathbf{1}_k$. The value of $\bar\beta$ that gives the local minimum of the effective action can be obtained by solving

$$\frac{\partial\Gamma_{1-\text{loop}}}{\partial\bar\beta} = N^2\left\{\frac{2}{k}(\bar\beta^3 - \bar\alpha\bar\beta^2) + \frac{6}{\bar\beta}\right\} = 0\,. \tag{C.33}$$

The solution exists for $\bar\alpha > \bar\alpha_{\text{cr}}^{(k\text{CP}^2)} = \frac{4}{\sqrt{3}}k^{\frac{1}{4}}$, and it is given explicitly as

$$\bar\beta = f(\bar\alpha) \equiv \frac{\bar\alpha}{4}\left(1 + \sqrt{1+\delta} + \sqrt{2 - \delta + \frac{2}{\sqrt{1+\delta}}}\right), \tag{C.34}$$

where

$$\delta = \bar\alpha^{-\frac{4}{3}}(96k)^{\frac{1}{3}}\left\{\left(1+\sqrt{1-\frac{256k}{9\bar\alpha^4}}\right)^{\frac{1}{3}} + \left(1-\sqrt{1-\frac{256k}{9\bar\alpha^4}}\right)^{\frac{1}{3}}\right\}. \tag{C.35}$$

At large $\bar\alpha$, the solution (C.34) can be expanded as

$$f(\bar\alpha) = \bar\alpha\left(1 - \sum_{j=1}^{\infty} c_j\bar\alpha^{-4j}\right) = \bar\alpha\left(1 - \frac{3k}{\bar\alpha^4} - \frac{27k^2}{\bar\alpha^8} - \frac{405k^3}{\bar\alpha^{12}} - \cdots\right). \tag{C.36}$$

Plugging this solution into (C.9), we obtain the free energy to all orders as

$$\frac{1}{N^2}W \simeq -\frac{\bar\alpha^4}{6k} + 6\log\bar\alpha + \log\frac{8N^{\frac{11}{2}}}{k^3} - \frac{9}{4} - \frac{9k}{\bar\alpha^4} - \frac{63k^2}{\bar\alpha^8} - \frac{1485k^3}{2\bar\alpha^{12}} - \cdots\,. \tag{C.37}$$

Using (C.29), (C.30) and (C.31), we obtain the all order results

$$\left\langle\frac{1}{N}\text{tr}(F_{\mu\nu})^2\right\rangle \simeq \frac{2\bar\alpha}{k}f(\bar\alpha)^3 + 8 = \frac{2\bar\alpha^4}{k} - 10 - \frac{108k}{\bar\alpha^4} - \frac{1512k^2}{\bar\alpha^8} - \frac{26730k^3}{2\bar\alpha^{12}} - \cdots\,. \tag{C.38}$$

$$\frac{1}{N^{\frac{1}{4}}}\langle M\rangle \simeq -\frac{2f(\bar\alpha)^3}{3k} = -\frac{2\bar\alpha^3}{3k} + \frac{6}{\bar\alpha} + \frac{36k}{\bar\alpha^5} + \frac{504k^2}{\bar\alpha^9} + \frac{8910k^3}{\bar\alpha^{13}} + \cdots\,. \tag{C.39}$$

$$\frac{1}{N^2}\langle S\rangle \simeq -\frac{1}{6k}\bar\alpha f(\bar\alpha)^3 + 2 = -\frac{\bar\alpha^4}{6k} + \frac{7}{2} + \frac{9k}{\bar\alpha^4} + \frac{126k^2}{\bar\alpha^8} + \frac{4455k^3}{2\bar\alpha^{12}} + \cdots\,. \tag{C.40}$$

Similarly to the case of the 3d YMCS model [66], we can also calculate various observables directly to all orders in perturbation theory in the large-N limit. For instance, the observable $\frac{1}{\sqrt{N}}\left\langle \frac{1}{N}\mathrm{tr}(A_\mu)^2 \right\rangle$ is obtained at one loop as (C.19), whose first and second terms correspond to the classical and one-loop contributions, respectively. As we see from (C.18), however, the one-loop contribution comes from the tadpole diagram, which is not one-particle irreducible. Therefore, by replacing $\bar{\alpha}$ by $f(\bar{\alpha})$ in the classical contribution, we obtain the all order result as

$$\frac{1}{\sqrt{N}}\left\langle \frac{1}{N}\mathrm{tr}(A_\mu)^2 \right\rangle \simeq \frac{2f(\bar{\alpha})^2}{3k} = \frac{2\bar{\alpha}^2}{3k} - \frac{4}{\bar{\alpha}^2} - \frac{30k}{\bar{\alpha}^6} - \frac{432k^2}{\bar{\alpha}^{10}} - \frac{7722k^3}{\bar{\alpha}^{14}} - \cdots . \tag{C.41}$$

D. Perturbative calculations for the fuzzy S^2

In this section we perform the perturbative analysis for the k coincident S^2 solution

$$X_\mu = A_\mu^{(k\,\mathrm{S}^2)} \equiv \begin{cases} \alpha\, L_\mu^{(n)} \otimes \mathbf{1}_k & \text{for } \mu = 1,2,3\,, \\ 0 & \text{otherwise}\,, \end{cases} \tag{D.1}$$

which generalizes (2.11). The total size of the matrix is now given by $N = nk$. Let us calculate the free energy as a perturbative expansion $W = \sum_{j=0}^{\infty} W_j$, where $W_j = \mathrm{O}(\alpha^{4(1-j)})$ comes from the j-loop contribution. From (B.13) the free energy is obtained at the one-loop level as

$$\begin{aligned} W_0 + W_1 &= -\frac{\alpha^4 N^2}{24}(n^2-1) + 3k^2 \sum_{l=1}^{n-1} (2l+1)\log[N\alpha^2 l(l+1)] - \log\{\mathrm{vol}(H)\mathcal{N}\} \\ &\simeq N^2 \left(-\frac{\tilde{\alpha}^4}{24k^2} + 6\log\tilde{\alpha} + \log\frac{N^7}{k^6} - \frac{15}{4} \right) , \end{aligned} \tag{D.2}$$

where the rescaled parameter $\tilde{\alpha}$ is defined by (4.1).

Similarly, the effective action for a one-parameter family of configurations

$$A_\mu = \begin{cases} \beta\, L_\mu^{(n)} \otimes \mathbf{1}_k & \text{for } \mu = 1,2,3\,, \\ 0 & \text{otherwise} \end{cases} \tag{D.3}$$

can be obtained at the one-loop level as

$$\Gamma_{1-\mathrm{loop}}(\tilde{\beta}) \simeq N^2 \left(\frac{\tilde{\beta}^4}{8k^2} - \frac{\tilde{\alpha}\tilde{\beta}^3}{6k^2} + 6\log\tilde{\beta} + \log\frac{N^7}{k^6} - \frac{15}{4} \right) , \tag{D.4}$$

where $\tilde{\beta} = \beta N^{\frac{1}{2}}$. The effective action has a local minimum

$$\begin{aligned} \tilde{\beta} &= g(\tilde{\alpha}) \equiv \frac{\tilde{\alpha}}{4}\left(1 + \sqrt{1+\varepsilon} + \sqrt{2 - \varepsilon + \frac{2}{\sqrt{1+\varepsilon}}} \right) , \\ \varepsilon &= \tilde{\alpha}^{-\frac{4}{3}} (384k^2)^{\frac{1}{3}} \left\{ \left(1 + \sqrt{1 - \frac{1024k^2}{9\tilde{\alpha}^4}} \right)^{\frac{1}{3}} + \left(1 - \sqrt{1 - \frac{1024k^2}{9\tilde{\alpha}^4}} \right)^{\frac{1}{3}} \right\} , \end{aligned} \tag{D.5}$$

if $\tilde{\alpha}$ is larger than the critical point

$$\tilde{\alpha}_{\rm cr}^{(k\,{\rm S}^2)} = \sqrt{\frac{32k}{3}} = 3.2659863\cdots \times \sqrt{k}\,. \tag{D.6}$$

Since the effective action is saturated at the one-loop level in the large-N limit [32], the critical point obtained above should be correct to all orders in perturbation theory.

The propagators for $\tilde{A}_\mu$ and the ghosts are exactly the same as [59, (C.1) and (C.2)]. The tadpole is given by

$$\langle \tilde{A}_\mu \rangle_{1-{\rm loop}} = -\frac{12k^2}{N^2\alpha^3} X_\mu\,. \tag{D.7}$$

The one-loop results for various observables are obtained as

$$\begin{aligned}
\frac{1}{N^{\frac{1}{2}}}\langle M\rangle_{1-{\rm loop}} &= -\frac{\alpha^3(n^2-1)}{6N^{\frac{1}{2}}} + \frac{6}{\alpha N^{\frac{1}{2}}}\left(1-\frac{1}{n^2}\right) \\
&\simeq -\frac{\tilde{\alpha}^3}{6k^2} + \frac{6}{\tilde{\alpha}}\,, && \text{(D.8)}\\
\left\langle \frac{1}{N}{\rm tr}(F_{\mu\nu})^2 \right\rangle_{1-{\rm loop}} &= 8\left(1-\frac{1}{N^2}\right) - 3\alpha\langle M\rangle_{1-{\rm loop}} \\
&= \frac{1}{2}\alpha^4(n^2-1) - 10 + \frac{1}{N^2}\left\{8(3k^2-1)-6k^2\right\} \\
&\simeq \frac{\tilde{\alpha}^4}{2k^2} - 10\,, && \text{(D.9)}\\
\frac{1}{N^2}\langle S\rangle_{1-{\rm loop}} &= \frac{1}{4}\left\langle \frac{1}{N}{\rm tr}(F_{\mu\nu})^2 \right\rangle_{1-{\rm loop}} + \alpha\langle M\rangle_{1-{\rm loop}} \\
&= -\frac{\alpha^4}{24}(n^2-1) + \frac{7}{2} + \frac{1}{2N^2}\left\{-4(k^2+1)+k^2\right\} \\
&\simeq -\frac{\tilde{\alpha}^4}{24k^2} + \frac{7}{2}\,, && \text{(D.10)}\\
\frac{1}{N}\left\langle \frac{1}{N}{\rm tr}(A_\mu)^2 \right\rangle_{1-{\rm loop}} &= \alpha^2\left\{\frac{1}{4N}(n^2-1) - \frac{6k^2(n^2-1)}{N^3\alpha^4} + \frac{8}{Nn^2\alpha^4}\sum_{l=1}^{n-1}\frac{2l+1}{l(l+1)}\right\} \\
&\simeq \tilde{\alpha}^2\left(\frac{1}{4k^2} - \frac{6}{\tilde{\alpha}^4} + \frac{16}{n^2\tilde{\alpha}^4}\log n\right) \\
&\simeq \tilde{\alpha}^2\left(\frac{1}{4k^2} - \frac{6}{\tilde{\alpha}^4}\right)\,. && \text{(D.11)}
\end{aligned}$$

Exploiting the one-loop saturation of the effective action in the large-N limit, we can calculate various quantities to all orders. The free energy is obtained as

$$\frac{1}{N^2}W \simeq -\frac{\tilde{\alpha}^4}{24k^2} + 6\log\tilde{\alpha} + \log\frac{N^7}{k^6} - \frac{15}{4} - \frac{36k^2}{\tilde{\alpha}^4} - \frac{1008k^4}{\tilde{\alpha}^8} - \frac{47520k^6}{\tilde{\alpha}^{12}} - \cdots \tag{D.12}$$

by plugging (D.5) into (D.4). Various observables are calculated as

$$\frac{1}{N^2}\langle S\rangle \simeq -\frac{1}{24k^2}\tilde{\alpha}\, g(\tilde{\alpha})^3+2=-\frac{\tilde{\alpha}^4}{24k^2}+\frac{7}{2}+\frac{36k^2}{\tilde{\alpha}^4}+\frac{2016k^4}{\tilde{\alpha}^8}+\frac{142560k^6}{\tilde{\alpha}^{12}}+\cdots, \tag{D.13}$$

$$\frac{1}{N}\left\langle \frac{1}{N}\mathrm{tr}(A_\mu)^2\right\rangle \simeq \frac{1}{4k^2}g(\tilde{\alpha})^2=\frac{\tilde{\alpha}^2}{4k^2}-\frac{6}{\tilde{\alpha}^2}-\frac{180k^2}{\tilde{\alpha}^2}-\frac{10368k^4}{\tilde{\alpha}^6}-\frac{741312k^6}{\tilde{\alpha}^{10}}-\cdots, \tag{D.14}$$

$$\frac{1}{N^{\frac{1}{2}}}\langle M\rangle \simeq -\frac{1}{6k^2}g(\tilde{\alpha})^3=-\frac{\tilde{\alpha}^3}{6k^2}+\frac{6}{\tilde{\alpha}}+\frac{144k^2}{\tilde{\alpha}^5}+\frac{8064k^4}{\tilde{\alpha}^9}+\frac{570240k^6}{\tilde{\alpha}^{13}}+\cdots, \tag{D.15}$$

$$\left\langle \frac{1}{N}\mathrm{tr}(F_{\mu\nu})^2\right\rangle \simeq \frac{1}{2k^2}\tilde{\alpha}\, g(\tilde{\alpha})^3+8=\frac{\tilde{\alpha}^4}{2k^2}-10-\frac{432k^2}{\tilde{\alpha}^4}-\frac{24192k^4}{\tilde{\alpha}^8}-\frac{1710720k^6}{\tilde{\alpha}^{12}}-\cdots. \tag{D.16}$$

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